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## Jacobi structures revisited

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### Abstract

Jacobi algebroids, i.e. graded Lie brackets on the Grassmann algebra associated with a vector bundle which satisfy a property similar to that of the Jacobi brackets, are introduced. They turn out to be equivalent to generalized Lie algebroids in the sense of Iglesias and Marrero. Jacobi bialgebroids are defined in the same manner. A lifting procedure of elements of this Grassmann algebra to multivector fields on the total space of the vector bundle which preserves the corresponding Lie brackets is developed. This gives the possibility of associating canonically a Lie algebroid with any local Lie algebra in the sense of Kirillov.

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### 1. Introduction

This paper was originated as an attempt to understand the Lie algebroid structure on  $T^*M \oplus_M \mathbb{R}$  associated with a Jacobi structure  $(\Lambda, \Gamma)$  on a manifold  $M$  [Li]. The formula in [KSB]

$$[(\alpha, f), (\beta, g)] = (\mathcal{L}_{\Lambda_\alpha}\beta - \mathcal{L}_{\Lambda_\beta}\alpha - d\langle\Lambda, \alpha \wedge \beta\rangle + f\mathcal{L}_\Gamma\beta - g\mathcal{L}_\Gamma\alpha \\ - i_\Gamma\alpha \wedge \beta, \langle\Lambda, \beta \wedge \alpha\rangle + \Lambda_\alpha(g) - \Lambda_\beta(f) + f\Gamma(g) - g\Gamma(f))$$

which gives the Lie bracket on the space  $\Omega^1(M) \times C^\infty(M)$  of sections of  $T^*M \oplus_M \mathbb{R}$ , being rather complicated, deserves a better understanding and explanation. During our work we have noticed that it is very close to [IM1] which will be our primary reference paper.

Since a Jacobi bracket is just a Lie bracket on the algebra of smooth functions given by a bilinear first-order differential operator, we start with the study of the Nijenhuis–Richardson bracket on multilinear first-order differential operators. This bracket is a graded Lie bracket

but it differs from the Richardson–Nijenhuis bracket. This is manifested by the fact that with respect to the wedge product it is not a derivation but a first-order differential operator. This is like the difference between the Poisson and Jacobi brackets.

We then discuss the case of a general Lie algebroid. Our primary object is the Schouten–Nijenhuis bracket associated with the Lie algebroid rather than the Lie algebroid bracket itself. Deforming the Schouten–Nijenhuis bracket to a graded Lie bracket which violates the Leibniz rule, like in the case of the Nijenhuis–Richardson bracket for first-order differential operators, we introduce the notion of a *Jacobi algebroid*. We find out that this is a structure equivalent to the notion of a *Lie algebroid with the presence of 1-cocycle* as defined in [IM1].

Since any Lie algebroid structure on a vector bundle  $E$  is associated with a linear Poisson structure on the dual bundle  $E^*$ , one can expect that there is a lifting procedure of multilinear first-order differential operators acting on smooth functions on  $M$  to multivector fields on  $TM \oplus_M \mathbb{R}$ , similar to the classical complete tangent lift of multivector fields on  $M$  (cf [IY, GU]), which associates the corresponding linear Poisson structure with a given Jacobi bracket. We define such lifts for Jacobi algebroids and show that the lift of a Jacobi structure gives exactly the Lie algebroid bracket (50). We extend this for general local Lie algebra structure in the sense of Kirillov [Ki]. The main result is that any local Lie algebra structure on a one-dimensional bundle  $L$  induces naturally a Lie algebroid structure on the first jet bundle  $J_1(L)$ .

Introducing a Cartan calculus for a given Jacobi Lie algebroid as in [IM1] one can define *Jacobi bialgebroids*, by analogy to the Lie bialgebroids, as Jacobi algebroid structures on dual pair of vector bundles such that the exterior differential induced by one structure is a graded derivation for the Schouten–Jacobi bracket of the second one. We show that this reduces exactly to the notion of a generalized Lie bialgebroid in [IM1]. The advantage of using consequently graded brackets on the corresponding Grassmann algebras is that this definition becomes more natural.

## 2. Graded Lie brackets

A *graded Lie bracket* on a graded vector space  $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n$  ('graded' means always ' $\mathbb{Z}$ -graded' throughout this paper) is a bilinear operation  $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ , being graded

$$[\mathcal{A}^n, \mathcal{A}^m] \subset \mathcal{A}^{n+m} \quad (1)$$

graded skew-symmetric

$$[X, Y] = -(-1)^{xy}[Y, X] \quad (2)$$

and satisfying the graded Jacobi identity

$$[[X, Y], Z] = [X, [Y, Z]] - (-1)^{xy}[Y, [X, Z]] \quad (3)$$

where we fix the convention that we write simply  $x, a$ , etc, for the Lie algebra degrees of homogeneous elements  $X, A$ , etc, when no confusion arises.

One sometimes writes the graded Jacobi identity in the form

$$(-1)^{xz}[[X, Y], Z] + (-1)^{yx}[[Y, Z], X] + (-1)^{zy}[[Z, X], Y] = 0 \quad (4)$$

which is equivalent to (3) for the graded skew-symmetric brackets. However, for the non-skew-symmetric brackets the formula (3) seems to be better, since it means that the adjoint

map  $X \mapsto \text{ad}_X \stackrel{\text{def}}{=} [X, \cdot]$  is a representation of the bracket, i.e.  $\text{ad}_{[X,Y]}$  is equal to the graded commutator

$$[\text{ad}_X, \text{ad}_Y] \stackrel{\text{def}}{=} \text{ad}_X \circ \text{ad}_Y - (-1)^{xy} \text{ad}_Y \circ \text{ad}_X = \text{ad}_{[X,Y]} \tag{5}$$

whereas (4) has no clear direct meaning.

With a given smooth ( $C^\infty$ ) manifold  $M$  several natural graded Lie brackets of tensor fields are associated. Historically the first one was probably the famous Schouten–Nijenhuis bracket  $[\cdot, \cdot]^{SN}$  defined on multivector fields (see [Sc, Ni]). It is the unique graded extension of the usual bracket  $[\cdot, \cdot]$  on the space  $\mathcal{X}(M)$  of vector fields to the exterior algebra  $A(M) = \bigoplus_{n \in \mathbb{Z}} A^{[n]}(M)$  of multivector fields (where  $A^{[n]}(M) = \Gamma(\Lambda^n T M)$  is the space of  $n$ -vector fields for  $n \geq 0$  and  $A^{[n]}(M) = \{0\}$  for  $n < 0$ ) such that

- (a) the degree of  $X \in A^{[n]}(M)$  with respect to the bracket is  $(n - 1)$
- (b)  $[X, f]^{SN} = X(f)$
- (c)  $[X, Y \wedge Z]^{SN} = [X, Y]^{SN} \wedge Z + (-1)^{(k-1)l} Y \wedge [X, Z]^{SN}$ , for  $X \in A^{[k]}(M)$ ,  $Y \in A^{[l]}(M)$ , i.e.  $\text{ad}$  is a representation of the Schouten–Nijenhuis bracket in graded derivations of the graded associative algebra  $A(M)$ .

The Schouten–Nijenhuis bracket is an example of what is sometimes called a *Gerstenhaber algebra* (see [KS, KS1]) which consists of a triple  $(\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n, \wedge, [\cdot, \cdot])$ , such that  $(\mathcal{A}, [\cdot, \cdot])$  is a graded Lie algebra and  $(\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^{[n]}, \wedge)$ , with  $\mathcal{A}^{[n]} = \mathcal{A}^{n+1}$ , is a graded associative commutative algebra, and  $\text{ad}_X$ , for  $\text{ad}_X \in \mathcal{A}^x$  is a derivation with respect to  $\wedge$  with degree  $x$ , i.e.

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{x(y+1)} Y \wedge [X, Z]. \tag{6}$$

From (6) it follows that in any Gerstenhaber algebra

$$\begin{aligned} & [X_1 \wedge \dots \wedge X_m, Y_1 \wedge \dots \wedge Y_n] \\ &= \sum_{k,l} (-1)^{k+l} [X_k, Y_l] \wedge \dots \wedge \hat{X}_k \wedge \dots \wedge X_m \wedge Y_1 \wedge \dots \wedge \hat{Y}_l \wedge \dots \wedge Y_n \end{aligned} \tag{7}$$

where  $X_k, Y_l \in \mathcal{A}^0$  and the hats stand for omissions. Note that  $\mathcal{A}^0$  is a Lie subalgebra of  $\mathcal{A}$  and  $V = \mathcal{A}^{-1}$  is an associative commutative subalgebra of  $\mathcal{A}$ .

The Schouten–Nijenhuis bracket is a particular case of the so-called *Nijenhuis–Richardson bracket* [NR] which, in turn, is the skew-symmetrization of the natural graded Lie bracket on multilinear operators discovered by Gerstenhaber [Ge]. We will call the last bracket the *Gerstenhaber bracket*. We will define all these brackets in detail to fix notation and signs.

*Gerstenhaber bracket.* Let  $V$  be a vector space over a field  $k$ . Denote the space of  $(p + 1)$ -linear maps  $A : V \times \dots \times V \rightarrow V$  by  $M^p(V)$ . On the graded vector space  $M(V) = \bigoplus_{p \in \mathbb{Z}} M^p(V)$ , where  $M^{-1}(V) = V$  and  $M^p(V) = \{0\}$  for  $p < -1$ , we define first *insertion operators*. For  $A \in M^a(V)$ ,  $B \in M^b(V)$ , the insertion  ${}_B A \in M^{a+b}(V)$  is defined by

$${}_B A(x_0, \dots, x_{a+b}) = \sum_{k=0}^a (-1)^{bk} A(x_0, \dots, x_{k-1}, B(x_k, \dots, x_{k+b}), x_{k+b+1}, \dots, x_{a+b}). \tag{8}$$

Then the *Gerstenhaber bracket*  $[A, B]^G$  is given by

$$[A, B]^G = -{}_A B + (-1)^{ab} {}_B A. \tag{9}$$

We can consider the graded subspace  $\mathcal{A}(V)$  of skew-symmetric elements of  $M(V)$ . The *Nijenhuis–Richardson bracket* on  $\mathcal{A}(V)$  is the skew-symmetrization of the Gerstenhaber bracket

$$[A, B]^{NR} = \frac{(a + b + 1)!}{(a + 1)!(b + 1)!} \text{skew}([A, B]^G) \tag{10}$$

where skew stands for the antisymmetrization projector in  $M(V)$ . We have  $[A, B]^{NR} = -i_A B + (-1)^{ab} i_B A$ , with

$$i_B A(x_0, \dots, x_{a+b}) = \sum_{\sigma \in S(a+b, b)} (-1)^\sigma A(B(x_{\sigma(0)}, \dots, x_{\sigma(b)}), x_{\sigma(b+1)}, \dots, x_{\sigma(a+b)}) \quad (11)$$

where  $S(a+b, b)$  is the set of unshuffles  $\sigma: \{0, \dots, a+b\} \rightarrow \{0, \dots, a+b\}$  with  $\sigma(0) < \dots < \sigma(b)$ ,  $\sigma(b+1) < \dots < \sigma(a+b)$ . The following is well known.

**Theorem 1.** *The Gerstenhaber bracket and the Nijenhuis–Richardson bracket make the graded vector spaces  $M(V)$  and  $\mathcal{A}(V)$ , respectively, into graded Lie algebras. Moreover,*

- (a) *the equation  $[A, A]^G = 0$  for  $A \in M^1(V)$  is equivalent to the fact that the bilinear operation  $A$  on  $V$  is associative;*
- (b) *the equation  $[A, A]^{NR} = 0$  for  $A \in \mathcal{A}^1(V)$  is equivalent to the fact that the skew-bilinear operation  $A$  on  $V$  is a Lie bracket.*

Suppose now that  $V$  is an associative commutative algebra with unit  $\mathbf{1}$ . Then  $M(V)$  is a graded associative algebra (with elements of  $M^a(V)$  being of degree  $a+1$ ) with the obvious product

$$A \cdot B(x_0, \dots, x_{a+b+2}) = A(x_0, \dots, x_a) B(x_{a+1}, \dots, x_{a+b+2}) \quad (12)$$

for  $A \in \mathcal{A}^a(V)$ ,  $B \in \mathcal{A}^b(V)$ . Similarly,  $\mathcal{A}(V)$  is, in a natural way, a graded associative commutative algebra (again, with elements of  $M^a(V)$  being of degree  $a+1$ ) with the obvious wedge product

$$A \wedge B = \frac{(a+b+2)!}{(a+1)!(b+1)!} \text{skew}(A \cdot B). \quad (13)$$

Denote by  $\text{Diff}(V)$ ,  $\text{Diff}_1(V)$  and  $\text{Der}(V)$ , respectively, the space of linear differential operators, linear-first order differential operators and derivations on  $V$ . Similarly, by  $\mathcal{ADiff}^p(V)$ ,  $\mathcal{ADiff}_1^p(V)$  and  $\mathcal{ADer}^p(V)$ , we denote the corresponding skew  $(p+1)$ -linear operators on  $V$  which are, respectively, differential operators, first-order differential operators and derivations with respect to each variable separately. By  $\mathcal{ADiff}(V)$ ,  $\mathcal{ADiff}_1(V)$  and  $\mathcal{ADer}(V)$ , we denote the corresponding graded vector spaces. It is easy to see the following (cf [Gr]):

**Theorem 2.**

- (a)  *$\mathcal{ADiff}(V)$ ,  $\mathcal{ADiff}_1(V)$  and  $\mathcal{ADer}(V)$  are graded associative subalgebras of  $(\mathcal{A}(V), \wedge)$  and graded Lie subalgebras of  $(\mathcal{A}(V), [\cdot, \cdot]^{NR})$ .*
- (b) *There is a canonical splitting*

$$\mathcal{ADiff}_1^p(V) = \mathcal{ADer}^p(V) \oplus \mathcal{ADer}^{p-1}(V) \quad (14)$$

*given by  $A = A_1 + I \wedge A_2$ , where  $A_1 \in \mathcal{ADer}^p(V)$ ,  $A_2 = i_1 A = A(\mathbf{1}, \cdot, \dots, \cdot) \in \mathcal{ADer}^{p-1}(V)$  and  $I$  is the identity map on  $V$ .*

- (c)  *$(\mathcal{ADer}(V), \wedge, [\cdot, \cdot]^{NR})$  is a Gerstenhaber algebra. In the case when  $V = C^\infty(M)$  is the algebra of smooth functions on a manifold  $M$ , one has  $\mathcal{ADer}(V) = \mathcal{A}(M)$  and the Nijenhuis–Richardson bracket reduces to the Schouten–Nijenhuis bracket.*

**Proof.** One can find the proofs of the parts (a) and (b) in [Gr], section 3. The part (c) follows easily from the following properties of the insertion operators versus wedge products:

$$i_C(A \wedge B) = (i_C A) \wedge B + (-1)^{c(a+1)} A \wedge (i_C B) \quad (15)$$

$$i_{A \wedge B} C = (-1)^{c(b+1)} (i_A C) \wedge B + A \wedge (i_B C) \quad \text{for } C \in \mathcal{ADer}^c(V). \quad (16)$$

Here, according to our convention,  $A \in \mathcal{A}^a(V)$ , i.e.  $A$  is  $(a+1)$ -linear, etc.  $\square$

**Remark.** Our convention of signs in the Gerstenhaber, and hence in the Nijenhuis–Richardson bracket, is different from the original one. This is chosen in this way in order to get a Gerstenhaber algebra structure on  $ADer(V)$  and hence on  $A(M)$ . Also the standard Schouten–Nijenhuis bracket, which is still used by many authors, differs by sign from ours. In particular, the Schouten bracket used in [IM1] is not a graded Lie bracket, since it is not graded skew-symmetric. It seems reasonable to use consequently graded Lie algebra brackets in order to avoid confusions. This will also simplify certain formulae and definitions, as we will see it later.

Since we already know that for an associative commutative algebra  $V$  the triple  $(ADer(V), \wedge, [\cdot, \cdot]^{NR})$  is a Gerstenhaber algebra, let us look closer at the structure of the algebra  $(ADiff_1(V), \wedge, [\cdot, \cdot]^{NR})$ . In the case  $V = C^\infty(M)$  we will write  $ADiff_1(M)$  instead of  $ADiff_1(V)$ .

Since  $i_{A \wedge B} I = A \wedge B = (i_A I) \wedge B + A \wedge (i_B I) - A \wedge B$ , we have for  $C \in ADiff_1^c(V)$  instead of (16) the following:

$$i_{A \wedge B} C = (-1)^{c(b+1)}(i_A C) \wedge B + A \wedge (i_B C) - A \wedge B \wedge i_1 C. \tag{17}$$

This, in turn, implies that on  $ADiff_1(V)$  we have

$$[A, B \wedge C]^{NR} = [A, B]^{NR} \wedge C + (-1)^{a(b+1)} B \wedge [A, C]^{NR} - (-1)^a i_1 A \wedge B \wedge C. \tag{18}$$

Note that  $D = i_1$  is a graded derivative of the wedge product of degree  $-1$  and  $\tilde{D}(X) = (-1)^x D(X)$  defines a right derivative:

$$\tilde{D}(X \wedge Y) = X \wedge \tilde{D}(Y) + (-1)^{y+1} \tilde{D}(X) \wedge Y. \tag{19}$$

In general, we will call the *Gerstenhaber–Jacobi algebra* a triple  $(\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n, \wedge, [\cdot, \cdot])$  as in the Gerstenhaber algebra case but with the graded bracket satisfying

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{x(y+1)} Y \wedge [X, Z] - \tilde{D}(X) \wedge Y \wedge Z \tag{20}$$

where  $\tilde{D}$  is a graded linear map of degree  $-1$ , instead of the Leibniz rule. Putting  $Y = Z = \mathbf{1}$  we obtain that  $\tilde{D}(X) = [X, \mathbf{1}]$ , so that  $\tilde{D}$  is a right-graded derivation of degree  $-1$  with respect to both: the associative and Lie algebra structures. Here we assume that the associative commutative algebra  $V = \mathcal{A}^{-1}$  has the unit  $\mathbf{1}$  (if not, we can always easily extend the whole structure). Thus, in the case of a Gerstenhaber–Jacobi algebra  $\text{ad}_X$  is not a derivative but a differential operator of first order with respect to the wedge product (cf [Ko]).

### 3. Lie algebroids and Jacobi algebroids

Let  $M$  be a smooth manifold. A *Lie algebroid on  $M$*  is a vector bundle  $\tau:L \rightarrow M$ , together with a bracket  $[\cdot, \cdot]:\Gamma L \times \Gamma L \rightarrow \Gamma L$  on the  $C^\infty(M)$ -module  $\Gamma L$  of smooth sections of  $L$ , and a  $C^\infty(M)$ -linear map  $a:\Gamma L \rightarrow \mathcal{X}(M)$  from  $\Gamma L$  to the Lie algebra of vector fields on  $M$ , called the *anchor* of the Lie algebroid, such that

- (i) the bracket on  $\Gamma L$  is  $\mathbb{R}$ -bilinear, alternating and satisfies the Jacobi identity.
- (ii)  $[X, fY] = f[X, Y] + a(X)(f)Y$  for all  $X, Y \in \Gamma L$  and all  $f \in C^\infty(M)$ .

From (i) and (ii) it follows easily

- (iii)  $a([X, Y]) = [a(X), a(Y)]$  for all  $X, Y \in \Gamma L$ .

We get an algebraic counterpart of the notion of Lie algebroid replacing the algebra  $C^\infty(M)$  of smooth functions by an arbitrary associative commutative algebra  $V$ , and the module of sections of the vector bundle  $\tau:L \rightarrow M$  by a module  $\mathcal{L}$  over the algebra  $V$ : a *Lie pseudoalgebra* over  $V$  is a  $V$ -module  $\mathcal{L}$  together with a bracket  $[\cdot, \cdot]:\mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$  on the module  $\mathcal{L}$ , and a  $V$ -module morphism  $a:\mathcal{L} \rightarrow \text{Der}(V)$  from  $\mathcal{L}$  to the  $V$ -module  $\text{Der}(V)$  of derivations of  $V$ , called the *anchor* of  $\mathcal{L}$ , such that

- (i) the bracket on  $\mathcal{L}$  is bilinear, alternating and satisfies the Jacobi identity;
- (ii) For all  $X, Y \in \mathcal{L}$  and all  $f \in V$  we have

$$[X, fY] = f[X, Y] + a(X)(f)Y \tag{21}$$

- (iii)  $a([X, Y]) = [a(X), a(Y)]$  for all  $X, Y \in \mathcal{L}$ .

As before, (i) and (ii) imply (iii) if only the  $V$ -module  $\mathcal{L}$  is faithful.

Lie algebroids on a singleton base space are Lie algebras. Another extreme example is the tangent bundle  $TM$  with the canonical bracket on the space  $\mathcal{X}(M) = \Gamma TM$  of vector fields.

Lie pseudoalgebras in slightly more general setting appeared first in the paper of Herz [He] but one can find similar concepts under more than a dozen of names in the literature (e.g.  $(R, A)$ -Lie algebras, Lie–Cartan pairs, Lie–Rinehart algebras, differential algebras, etc). Lie algebroids were introduced by Pradines [Pr]. For both notions we refer to a survey article by Mackenzie [Ma].

From now on we assume that  $L$  is a vector bundle over  $M$ ,  $V(M) = C^\infty(M)$  is the algebra of smooth functions on  $M$ ,  $\mathcal{L}$  is the  $V(M)$ -module of smooth sections of  $L$ . Any Gerstenhaber algebra structure on the Grassmann algebra  $\mathcal{A}(L) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n(L)$ , where  $\mathcal{A}^n(L) = \Gamma(\bigwedge^{n+1} L)$ , we will call a *Schouten–Nijenhuis algebra*. As it was already indicated in [KS], the Schouten–Nijenhuis algebras are in one–one correspondence with the Lie algebroids:

**Theorem 3.** *Any Schouten–Nijenhuis bracket  $[\cdot, \cdot]$  on  $\mathcal{A}(L)$  induces a Lie algebroid bracket on  $\mathcal{L} = \mathcal{A}^0(L)$  with the anchor defined by  $a(X)(f) = [X, f]$ . Conversely, any Lie algebroid structure on  $\mathcal{L}$  gives rise to a Schouten–Nijenhuis bracket on  $\mathcal{A}(L)$  for which  $\mathcal{L} = \mathcal{A}^0(L)$  is a Lie subalgebra and  $a(X)(f) = [X, f]$ .*

Let  $\Omega(L) = \bigoplus_{n \in \mathbb{Z}} \Omega^{[n]}$  be the  $V(M)$ -module dual to  $\mathcal{A}(L) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^{[n]}(L)$ , where  $\Omega^{[n]}(L) = \Gamma(\bigwedge^n L^*)$  is the space of sections of the  $n$ -exterior power of the bundle  $L^*$  dual to  $L$ . We can think of elements of  $\mathcal{A}^{[n]}(L)$  as being ‘ $n$ -vector fields’ and elements of  $\Omega^{[n]}(L)$  as being ‘ $n$ -forms’.

The Lie algebroid bracket on  $\mathcal{L} = \Gamma L$  induces the well-known generalization of the standard Cartan calculus of differential forms and vector fields [Ma, MX] (one can find an algebraic calculus for the gauge theories built on extensions of the Lie algebroids in [LM]).

The exterior derivative  $d: \Omega^{[k]}(L) \rightarrow \Omega^{[k+1]}(L)$  is defined by the standard formula

$$d\mu(X_1, \dots, X_{k+1}) = \sum_i (-1)^{i+1} [X_i, \mu(X_1, \dots, \hat{X}_i, \dots, X_{k+1})] + \sum_{i < j} (-1)^{i+j} \mu([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}) \tag{22}$$

where  $X_i \in \mathcal{L}$  and the hat over a symbol means that this is to be omitted. For  $X \in \mathcal{L}$ , the contraction  $i_X: \Omega^{[p]}(L) \rightarrow \Omega^{[p-1]}(L)$  is defined in the standard way and the Lie differential operator  $\mathfrak{L}_X$  is defined by the graded commutator

$$\mathfrak{L}_X = i_X \circ d - d \circ i_X. \tag{23}$$

The following theorem contains a list of well-known properties of these objects.

**Theorem 4.** *Let  $\mu \in \Omega^{[k]}(L)$ ,  $\nu \in \Omega(L)$  and  $X, Y \in \mathcal{L}$ . We have*

- (a)  $d \circ d = 0$
- (b)  $d(\mu \wedge \nu) = d(\mu) \wedge \nu + (-1)^k \mu \wedge d(\nu)$
- (c)  $\mathfrak{L}_X(\mu \wedge \nu) = \mathfrak{L}_X \mu \wedge \nu + \mu \wedge \mathfrak{L}_X \nu$
- (d)  $\mathfrak{L}_X \circ \mathfrak{L}_Y - \mathfrak{L}_Y \circ \mathfrak{L}_X = \mathfrak{L}_{[X, Y]}$
- (e)  $\mathfrak{L}_X \circ i_Y - i_Y \circ \mathfrak{L}_X = i_{[X, Y]}$ .

Let us now consider the Gerstenhaber–Jacobi structures on the Grassmann algebra  $\mathcal{A}(L)$ . We will call these *Schouten–Jacobi algebras*. In view of the previous theorem, we will identify the Schouten–Jacobi brackets on  $\mathcal{A}(L)$  with *Jacobi algebroid* structures on  $L$ . We will see that a Jacobi algebroid structure on  $L$  is determined by a Lie algebroid structure on  $L$  and a ‘1-form’  $\Phi \in \Omega^{[1]}(L)$  which is closed, i.e.  $d\Phi = 0$ . Indeed, since  $D(X) = (-1)^x \tilde{D}(X)$  defines a graded derivative of the wedge product of degree  $-1$ ,  $D$  is  $V(M)$ -linear, so  $D = i_\Phi$  for some  $\Phi \in \Omega^{[1]}(L)$ . Moreover, since  $\tilde{D}(X) = [X, \mathbf{1}]$ , the graded Jacobi identity implies that  $D$  is a derivative also for the Lie bracket, i.e.  $D([X, Y]) = [D(X), Y] + [X, D(Y)]$  for  $X, Y \in \mathcal{A}^0(L)$  which means exactly that  $d\Phi = 0$ . Further, the Schouten–Jacobi bracket restricted to  $\mathcal{A}^0(L)$  is a Lie algebroid bracket. Indeed, (20) implies, in particular, that  $[X, fg] = [X, f]g + f[X, g] - D(X)fg$  for  $X \in \mathcal{A}^0(L)$ ,  $f \in V(M)$ , which means that  $\text{ad}_X$  induces on  $V$  a first-order differential operator  $\hat{X} + D(X)I$ , where  $\hat{X}$  is a derivation and  $D(X) = \text{ad}_X(\mathbf{1})$  is the part of order 0 (cf theorem 2). Since for  $X, Y \in \mathcal{A}^0$ ,  $f \in V$ , we have in view of (20)

$$[X, fY] = ([X, f] - D(\mathbf{1})f)Y + f[X, Y] = \hat{X}(f)Y + f[X, Y] \tag{24}$$

the bracket on  $\mathcal{A}^0(L)$  is a Lie algebroid bracket with the anchor  $a(X) = \hat{X}$ .

Conversely, having a Lie algebroid bracket  $[\cdot, \cdot]$  on  $\mathcal{A}^0(L)$  with an anchor  $a$  and a closed ‘1-form’  $\Phi \in \Omega^{[1]}(L)$  we can construct a Schouten–Jacobi bracket on  $\mathcal{A}(L)$  as it was done in [IM1] (but here the signs are adapted to our conventions):

$$[X, Y]^\Phi = [X, Y]^{SN} + xX \wedge i_\Phi Y - (-1)^x y i_\Phi X \wedge Y \tag{25}$$

where  $[\cdot, \cdot]^{SN}$  is the Schouten–Nijenhuis bracket on  $\mathcal{A}(L)$  induced by the Lie algebroid structure and  $x, y$  are Lie algebra degrees of  $X, Y$ . This bracket gives the original Lie algebroid structure on  $\mathcal{A}^0$  and  $\tilde{D}(X) = [X, \mathbf{1}] = (-1)^x i_\Phi X$  for  $X \in \mathcal{A}(L)$ . Thus we get the following.

**Theorem 5.** *The formula (25) describes a one–one correspondence between Schouten–Jacobi brackets on  $\mathcal{A}(L)$  and Jacobi algebroid structures on  $L$ , i.e. Lie algebroid brackets on  $\mathcal{A}^0(L)$  defining Schouten–Nijenhuis brackets  $[\cdot, \cdot]^{SN}$  with the presence of a 1-cocycle  $\Phi \in \Omega^{[1]}(L)$ ,  $d\Phi = 0$ .*

One can develop a Cartan calculus for the Jacobi algebroids similarly to the Lie algebroid case (cf [IM1]). For a Schouten–Jacobi bracket associated with a 1-cocycle  $\Phi$  the definitions of the exterior differential  $d^\Phi$  and Lie differential  $\mathfrak{L}^\Phi = d^\Phi \circ i + i \circ d^\Phi$  are formally the same as (22) and (23), respectively. Since, for  $X \in \mathcal{A}^0(L)$ ,  $f \in V(M)$ , we have  $[X, f] = [X, f]^{SN} + (i_\Phi X)f$ , one obtains  $d^\Phi \mu = d\mu + \Phi \wedge \mu$ . Here  $[\cdot, \cdot]^{SN}$  and  $d$  are, respectively, the Schouten–Nijenhuis bracket and the exterior derivative associated with the Lie algebroid of the given Jacobi algebroid. For the exterior differential and Lie differential associated with a Jacobi algebroid we have the following.

**Theorem 6.** *Let  $\mu \in \Omega^{[k]}(L)$ ,  $\nu \in \Omega(L)$  and  $X, Y \in \mathcal{L}$ . We have*

- (a)  $d^\Phi \circ d^\Phi = 0$
- (b)  $d^\Phi(\mu \wedge \nu) = d^\Phi(\mu) \wedge \nu + (-1)^k \mu \wedge d^\Phi(\nu) - \Phi \wedge \mu \wedge \nu$
- (c)  $\mathfrak{L}_X^\Phi(\mu \wedge \nu) = \mathfrak{L}_X^\Phi \mu \wedge \nu + \mu \wedge \mathfrak{L}_X^\Phi \nu - (i_\Phi X)\mu \wedge \nu$
- (d)  $\mathfrak{L}_X^\Phi \circ i_Y - i_Y \circ \mathfrak{L}_X^\Phi = i_{[X, Y]}$
- (e)  $\mathfrak{L}_X^\Phi \circ \mathfrak{L}_Y^\Phi - \mathfrak{L}_Y^\Phi \circ \mathfrak{L}_X^\Phi = \mathfrak{L}_{[X, Y]}^\Phi$ .

**Proof.** The proof of (a), (b) and (c) can be found in [IM1]. The property (d) follows easily from definitions, and (e) follows easily from (d). □



**Example.** Consider the Jacobi algebroid structure associated with the Nijenhuis–Richardson bracket on  $\mathcal{ADiff}_1(M)$ . According to theorem 2,  $\mathcal{ADiff}_1^0(M)$  can be identified with  $\mathcal{X}(M) \oplus C^\infty(M)$ , i.e. with sections of the direct sum bundle  $TM \oplus_M \mathbb{R}$ . It is easy to see that the Lie algebroid bracket on this bundle reads (cf [IM1])

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)) \tag{26}$$

where the right-hand side bracket is the standard bracket of vector fields. The 1-cocycle  $\Phi$ , written as  $i_1$  in theorem 2, is given by  $\Phi(X, f) = f$ . The Schouten–Jacobi bracket (i.e. the Nijenhuis–Richardson bracket in this case) reads

$$[A_1 + I \wedge A_2, B_1 + I \wedge B_2]^{RN} = [A_1, B_1]^{SN} + (-1)^a I \wedge [A_1, B_2]^{SN} + I \wedge [A_2, B_1]^{SN} + aA_1 \wedge B_2 - (-1)^a bA_2 \wedge B_1 + (a - b)I \wedge A_2 \wedge B_2. \tag{27}$$

Hence, the bracket  $\{\cdot, \cdot\}$  on  $C^\infty(M)$  defined by a bilinear differential operator  $\Lambda + I \wedge \Gamma \in \mathcal{ADiff}_1^1(M)$  is a Lie bracket (Jacobi bracket on  $C^\infty(M)$ ) if and only if

$$[\Lambda + I \wedge \Gamma, \Lambda + I \wedge \Gamma]^{RN} = [\Lambda, \Lambda]^{SN} + 2I \wedge [\Gamma, \Lambda]^{SN} + 2\Lambda \wedge \Gamma = 0. \tag{28}$$

We recognize the conditions

$$[\Gamma, \Lambda]^{SN} = 0 \quad [\Lambda, \Lambda]^{SN} = -2\Lambda \wedge \Gamma \tag{29}$$

defining a Jacobi structure on  $M$  [Li]. The difference in the sign when comparing with [Li] comes from our convention for the Schouten bracket.

It is obvious that the formula (27) defines a Schouten–Jacobi bracket for any extension  $L \oplus_M \mathbb{R}$  of a Lie algebroid  $L$  associated with the anchor map

$$[(X, f), (Y, g)] = ([X, Y], a(X)(g) - a(Y)(f)). \tag{30}$$

The 1-cocycle is in this case  $\Phi((X, f)) = f$ .

Suppose that we have a Schouten–Jacobi bracket as above and  $Z \in \mathcal{A}^0(L)$ . We call  $X \in \mathcal{A}^x(L \oplus \mathbb{R})$  a  $Z$ -homogeneous element if  $[Z, X]^\Phi = -xX$ . The graded subspace spanned by  $Z$ -homogeneous elements is clearly a Lie subalgebra of the Schouten–Jacobi bracket. We can represent the Schouten–Jacobi bracket for  $Z$ -homogeneous elements of  $\mathcal{A}(L \oplus \mathbb{R})$  in the Schouten–Nijenhuis bracket of  $\mathcal{A}(L)$ .

**Theorem 7.** Let  $H_Z$  be the mapping which associated with any  $Z$ -homogeneous element  $A = A_1 + I \wedge A_2 \in \mathcal{A}^a(L \oplus \mathbb{R})$  the element  $H_Z(A) = A_1 + Z \wedge A_2$ . Then  $H_Z$  is a homomorphism of the Schouten–Jacobi bracket (27) on  $Z$ -homogeneous elements into the Schouten–Nijenhuis bracket on  $\mathcal{A}(L)$ :

$$\begin{aligned} [H_Z(A), H_Z(B)]^{SN} &= [A_1 + Z \wedge A_2, B_1 + Z \wedge B_2]^{SN} \\ &= [A_1, B_1]^{SN} + aA_1 \wedge B_2 - (-1)^a bA_2 \wedge B_1 \\ &\quad + (-1)^a Z \wedge [A_1, B_2]^{SN} + Z \wedge [A_2, B_1]^{SN} + (a - b)Z \wedge A_2 \wedge B_2 \\ &= H_Z([A, B]^\Phi). \end{aligned} \tag{31}$$

**4. Lifts of Schouten–Jacobi brackets**

Since in (25) we can put  $h\Phi$  instead of  $\Phi$ , where  $h$  is a parameter, the Schouten–Jacobi bracket can be viewed as a deformation of the Schouten–Nijenhuis bracket.

**Theorem 8.** The Schouten–Jacobi brackets  $[\cdot, \cdot]^{SN}$  and  $[\cdot, \cdot]_0^\Phi$  defined by

$$[X, Y]_0^\Phi = xX \wedge i_\Phi Y - (-1)^x y i_\Phi X \wedge Y \tag{32}$$

are compatible, i.e.  $[X, Y]^{h\Phi} = [X, Y]^{SN} + h[X, Y]_0^\Phi$  is a graded Lie bracket for all  $h \in \mathbb{R}$ .

An easy way to see that the Schouten–Jacobi bracket (25) is really a graded Lie bracket is the following.

Consider the product  $\tilde{L}$  of the Lie algebroid  $L$  and  $T\mathbb{R}$ , i.e. we view  $\tilde{L} = L \times T\mathbb{R}$  as a vector bundle over  $M \times \mathbb{R}$  with the obvious product Lie bracket and the anchor  $a \times \text{id}: L \times T\mathbb{R} \rightarrow TM \times T\mathbb{R}$ . For a fixed 1-cocycle  $\Phi \in \Omega^{[1]}(L)$  we define a Lie algebroid injective homomorphism  $U_\Phi : L \rightarrow \tilde{L}$  by

$$U_\Phi(X) = X + i_\Phi X \partial_t. \tag{33}$$

Here sections of  $L$  and functions on  $M$  on the right-hand side are understood as sections of  $L \times T\mathbb{R}$  and functions on  $M \times \mathbb{R}$  in an obvious way and  $\partial_t$  is the basic vector field on  $\mathbb{R}$ . Since  $\Phi$  is a 1-cocycle,

$$[U_\Phi(X), U_\Phi(Y)] = [X, Y] + ([X, i_\Phi Y] + [i_\Phi X, Y])\partial_t = [X, Y] + i_\Phi[X, Y]\partial_t = U_\Phi([X, Y]) \tag{34}$$

for  $X, Y \in \mathcal{L}$  and  $U_\Phi$  is really a homomorphism. This homomorphism can be extended to a homomorphism of the whole Gerstenhaber algebra by

$$U_\Phi(X) = X + \partial_t \wedge i_\Phi X \tag{35}$$

since  $U_\Phi$  respects the wedge product. Thus

$$[U_\Phi(X), U_\Phi(Y)]^{SN} = U_\Phi([X, Y]^{SN}) \tag{36}$$

for all  $X, Y \in \mathcal{A}(L)$ . Now, we can define a new graded linear map  $\tilde{U}_\Phi: \mathcal{A}(L) \rightarrow \mathcal{A}(\tilde{L})$  by  $\tilde{U}_\Phi(X) = e^{-Xt} U_\Phi(X)$ . This mapping respects grading but not the wedge product. However, the image of  $\tilde{U}_\Phi$  is a Lie subalgebra of the Schouten–Nijenhuis bracket on  $\mathcal{A}(\tilde{L})$ . It is easy to see the following.

**Theorem 9.** For  $X, Y \in \mathcal{A}(L)$  we have

$$[\tilde{U}_\Phi(X), \tilde{U}_\Phi(Y)]^{SN} = \tilde{U}_\Phi([X, Y]^{SN} + xX \wedge i_\Phi Y - (-1)^x y i_\Phi X \wedge Y). \tag{37}$$

Thus  $\tilde{U}_\Phi$  is an embedding of the Schouten–Jacobi bracket  $[\cdot, \cdot]^\Phi$  on  $\mathcal{A}(L)$  (induced by the Schouten–Nijenhuis bracket and the 1-cocycle  $\Phi$ ) into the Schouten–Nijenhuis bracket on  $\mathcal{A}(\tilde{L})$ , so (25) is a graded Lie bracket.

Note that on a similar idea is based the Poissonization of a Jacobi structure in [GL] and the construction a Lie algebroid from a Jacobi structure in [Va]. The bundle projection of  $L \times T\mathbb{R}$  over  $M \times \mathbb{R}$  onto  $L \times \mathbb{R}$  over  $M \times \mathbb{R}$  defines a Lie algebroid bracket on  $L \times \mathbb{R}$ , i.e. on ‘time-dependent sections of  $L$ ’ as described in [IM1]. Composing  $\tilde{U}_\Phi$  with this projection we get a representation of the Schouten–Jacobi bracket  $[\cdot, \cdot]^\Phi$  on  $\mathcal{A}(L)$  in the Schouten–Nijenhuis bracket of the Lie algebroid  $L \times \mathbb{R}$  (cf [IM1], section 4.2). The advantage of this construction is that the dimension of the fibres remains the same. On the other hand, in our construction the Lie algebroid is fixed and only the embedding depends on  $\Phi$ .

There is another approach to the Lie algebroids. As it was shown in [GU1, GU2], a Lie algebroid structure (or the corresponding Schouten–Nijenhuis bracket) is determined by the algebroid lift  $X \mapsto X^c$  which associates with  $X \in \mathcal{A}(L)$  a multivector field  $X^c \in A(M)$ . Recall that sections  $\mu$  of the dual bundle  $L^*$  may be identified with linear (along fibres) functions  $\iota_\mu$  on  $L$ :  $\iota_\mu(X_p) = \langle \mu(p), X_p \rangle$ . By *homogeneous elements* of the Schouten algebra of multivector fields on a vector bundle we understand elements which are homogeneous with respect to the Liouville vector field  $\nabla$ . This means that each contraction with differentials of linear functions  $\langle \Lambda, d\iota_{\mu_0} \wedge \cdots \wedge d\iota_{\mu_\Lambda} \rangle$  is again a linear function associated with an element

$[\mu_0, \dots, \mu_\lambda]_\Lambda$ . We call the *bracket induced by  $\Lambda$*  the multilinear operation  $[\mu_0, \dots, \mu_\lambda]_\Lambda$  on sections of  $L^*$ . Note that these brackets have a property similar to the Lie algebroid brackets:

$$[\mu_0, \dots, \mu_{\lambda-1}, f\mu_\lambda] = f[\mu_0, \dots, \mu_{\lambda-1}, \mu_\lambda] + \Lambda_{\mu_0, \dots, \mu_{\lambda-1}}(f)\mu_\lambda \tag{38}$$

where

$$\Lambda_{\mu_0, \dots, \mu_{\lambda-1}}(f) = [\mu_0, \dots, \mu_{\lambda-1}, \mathbf{d}f]_\Lambda \tag{39}$$

defines the *Hamiltonian vector field* of  $\Lambda$  associated with  $(\mu_0, \dots, \mu_{\lambda-1})$ . Thus, there is a one–one correspondence between linear multivector fields and such brackets.

**Theorem 10.** ([GU1]) *For a given Lie algebroid structure on a vector bundle  $L$  over  $M$  there is a unique complete lift of elements  $X$  of the Gerstenhaber algebra  $\mathcal{A}(L)$  to homogeneous elements  $X^c$  of the Schouten algebra  $A(L)$  of multivector fields on  $L$ , such that*

- (a)  $f^c = \iota_{\mathbf{d}f}$  for  $f \in C^\infty(M)$
- (b)  $X^c(\iota_\mu) = \iota_{\mathbb{E}_X \mu}$  for  $X \in \Gamma L$ ,  $\mu \in \Gamma L^*$
- (c)  $(X \wedge Y)^c = X^c \wedge Y^v + X^v \wedge Y^c$ , where  $X \mapsto X^v$  is the standard vertical lift of multisections of  $L$  to multivector fields on  $L$ .

Moreover, this complete lift is a homomorphism of the Schouten–Nijenhuis brackets:

$$[X, Y]^c = [X^c, Y^c] \tag{40}$$

and

$$[X^c, Y^v] = [X, Y]^v. \tag{41}$$

**Remark.** For the canonical Lie algebroid  $L = TM$ , the above complete lift reduces to the better-known *tangent lift* of the multivector fields on  $M$  to the multivector fields on  $TM$  (cf [IY, GU]). The complete Lie algebroid lift of just sections of  $L$ , i.e. the formula (b), was already indicated in [MX1].

Our aim is to find an analogue of the Lie algebroid complete lift for the Jacobi algebroids which will represent the Schouten–Jacobi bracket on  $\mathcal{A}(L)$  in the Nijenhuis–Richardson bracket of first-order multidifferential operators on  $L$ . Let  $[\cdot, \cdot]^\Phi$  be the Schouten–Jacobi bracket on  $\mathcal{A}(L)$  associated with a Lie algebroid structure on  $L$  and a 1-cocycle  $\Phi$ .

**Definition.** *The Jacobi lift of an element  $X \in \mathcal{A}^x(L)$  is the element  $\hat{X}_\Phi \in \mathcal{ADiff}_1(L)$ , i.e. a multidifferential operator of first order on  $L$ , defined by*

$$\hat{X}_\Phi = X^c - x \iota_\Phi X^v + I \wedge (i_\Phi X)^v \tag{42}$$

where  $X^c$  is the complete Lie algebroid lift and  $X^v$  is the vertical lift.

**Theorem 11.** *The Jacobi lift has the following properties:*

- (a)  $\hat{f}_\Phi = \iota_{\mathbf{d}^\Phi f}$  for  $f \in C^\infty(M)$
- (b)  $\hat{X}_\Phi(\iota_\mu) = \iota_{\mathbb{E}_X^\Phi \mu}$  and  $X_\Phi(\mathbf{1}) = \Phi(X) \circ \tau$  for  $X \in \Gamma L$ ,  $\mu \in \Gamma L^*$  and  $\tau: L \rightarrow M$  being the bundle projection
- (c)  $(X \wedge Y)_\Phi^\wedge = \hat{X}_\Phi \wedge Y^v + X^v \wedge \hat{Y}_\Phi - \iota_\Phi(X^v \wedge Y^v)$
- (d)  $[\hat{X}_\Phi, \hat{Y}_\Phi]^{NR} = ([X, Y]^\Phi)_\Phi$ .

**Proof.** The proof consists of standard calculations using the properties of the Schouten–Nijenhuis and Schouten–Jacobi brackets and the properties of the complete lift. One should also remember that  $i_\Phi[X, Y]^\Phi = [i_\Phi X, Y]^\Phi + (-1)^x[X, i_\Phi Y]^\Phi$  and use the identities (cf [GU1])  $[\iota_\Phi, X^v]^{SN} = -(i_\Phi X)^v$  and  $[\iota_\Phi, X^c]^{SN} = -(i_\Phi X)^c$  (the last one depends on  $\mathbf{d}\Phi = 0$ ). □

**Corollary 1.** *If  $X = \Lambda + I \wedge \Gamma$  is a Jacobi structure on  $M$ , then the Jacobi lift  $\hat{X}_\Phi$  is a homogeneous Jacobi structure on  $TM \oplus \mathbb{R}$ . Moreover,*

$$\hat{X}_\Phi = \Lambda^c + \partial_t \wedge \Gamma^c - t(\Lambda^v + \partial_t \wedge \Gamma^v) + I \wedge \Gamma^v \tag{43}$$

where  $\Lambda^c, \Lambda^v, \dots$  are the complete and vertical lifts to  $TM$  and  $t$  is the canonical linear coordinate in  $\mathbb{R}$ .

**Proof.** In our case the Lie algebroid is the extension  $TM \oplus \mathbb{R}$  relative to the anchor map and the complete and vertical lifts of  $\Lambda$  and  $\Gamma$  with respect to this Lie algebroid structure coincide with the standard tangent complete and vertical lifts. Moreover,  $I^c = 0, I^v = \partial_t$  and  $i_\Phi X = \Gamma$ . □

**Remark.** We get the same homogeneous Jacobi structure as [IM], example 5.

According to theorem 7, there is a homomorphism  $H_\nabla: ADiff_1(L) \rightarrow \mathcal{A}(L)$  of the Nijenhuis–Richardson bracket on homogeneous first-order multidifferential operators into the Schouten–Nijenhuis bracket of multivector fields on  $L$  given by  $H_\nabla(X_1 + I \wedge X_2) = X_1 + \nabla \wedge X_2$ , where  $\nabla$  is the Liouville vector field on the vector bundle  $L$ .

**Definition.** *The Poisson lift  $X_\Phi^c$  is defined by*

$$X_\Phi^c = H_\nabla(\hat{X}_\Phi) = X^c - x \iota_\Phi X^v + \nabla \wedge (i_\Phi X)^v. \tag{44}$$

**Theorem 12.** *The Poisson lift has the following properties:*

- (a)  $f_\Phi^c = \iota_{d^\Phi} f$  for  $f \in C^\infty(M)$
- (b)  $X_\Phi^c(\iota_\mu) = \iota_{x_\mu^\Phi}$
- (c)  $(X \wedge Y)_\Phi^c = X_\Phi^c \wedge Y^v + X^v \wedge Y_\Phi^c - \iota_\Phi(X^v \wedge Y^v)$
- (d)  $[X_\Phi^c, Y_\Phi^c]^{SN} = ([X, Y]^\Phi)^c$ .

**5. Lie algebroids associated with local Lie algebras**

It is known [Fu, Ko, GU], that a Poisson structure  $\Lambda$  on  $M$  defines not only the Poisson bracket  $\{\cdot, \cdot\}_\Lambda$  of functions, but also a Lie bracket  $[\cdot, \cdot]_\Lambda$  on 1-forms, given by

$$[\mu, \nu]_\Lambda = \mathfrak{L}_{\Lambda_\mu} \nu - \mathfrak{L}_{\Lambda_\nu} \mu - d\langle \Lambda, \mu \wedge \nu \rangle \tag{45}$$

where  $\Lambda_\mu = i_\mu \Lambda$  and  $\langle \cdot, \cdot \rangle$  is the pairing between forms and multivector fields. In particular,  $[df, dg]_\Lambda = d\{f, g\}_\Lambda$  and  $\Lambda^\#$  is a Lie bracket homomorphism:

$$[\Lambda_\mu, \Lambda_\nu]_\Lambda = \Lambda_{[\mu, \nu]_\Lambda}. \tag{46}$$

This bracket on 1-forms is a Lie algebroid bracket and it induces the corresponding Schouten–Nijenhuis bracket on  $\Omega(M)$ . It was observed by Koszul [Ko] (see also [KSM]) that this bracket has a *generating operator*  $\partial_\Lambda$ :

$$[\mu, \nu]_\Lambda = (-1)^m (\partial_\Lambda(\mu \wedge \nu) - \partial_\Lambda \mu \wedge \nu - (-1)^m \mu \wedge \partial_\Lambda \nu) \tag{47}$$

where  $\partial_\Lambda = i(P) \circ d - d \circ i(P)$  and  $m$  is the standard degree of the form  $\mu$ . Note that  $\partial_\Lambda$  is a homology operator, since  $\partial_\Lambda^2 = 0$ . Moreover,

$$d[\mu, \nu]_\Lambda = [d\mu, \nu]_\Lambda + (-1)^{m-1} [\mu, d\nu]_\Lambda. \tag{48}$$

The whole structure, i.e. the Gerstenhaber algebra with the generating operator  $\partial$  for the Lie bracket satisfying  $\partial^2 = 0$  is called a *Batalin–Vilkovisky algebra* (cf [KS]). With the presence

of the derivation  $d$  of both: the associative and Lie algebra structures it is a *differential Batalin–Vilkovisky algebra*.

It is well known [GU] that the Lie algebroid bracket (45) is induced by the complete lift  $\Lambda^c$ . Now, it should be no surprise that the Lie algebroid bracket (1) we started with is induced by our Poisson lift of the corresponding Jacobi structure.

**Theorem 13.** *If  $X = \Lambda + I \wedge \Gamma$  is a Jacobi structure on  $M$ , then the Poisson lift  $X_\Phi^c$  is a homogeneous Poisson structure on  $TM \oplus \mathbb{R}$ . Moreover,*

$$X_\Phi^c = \Lambda^c + \partial_t \wedge \Gamma^c - t\Lambda^v + \nabla \wedge \Gamma^v \tag{49}$$

where  $\Lambda^c, \Lambda^v$ , etc, are the complete and vertical lifts to  $TM$ , respectively,  $\nabla$  is the Liouville vector field on  $TM$  and  $t$  is the canonical linear coordinate in  $\mathbb{R}$ . This homogeneous Poisson structure determines a Lie algebroid bracket on the dual bundle  $T^*M \oplus \mathbb{R}$ , given for  $\mu, \nu \in \Gamma(T^*M \oplus \mathbb{R})$ ,  $\mu = (\alpha, f)$ ,  $\nu = (\beta, g)$ , by

$$[\mu, \nu] = \mathfrak{L}_{X_\mu}^\Phi \nu - \mathfrak{L}_{X_\nu}^\Phi \mu - d^\Phi \langle X, \mu \wedge \nu \rangle \tag{50}$$

where  $X_\mu = i_\mu X = (i_\alpha \Lambda + f\Gamma, -i_\alpha \Gamma)$ ,  $X_\nu = i_\nu X = (i_\beta \Lambda + g\Gamma, -i_\beta \Gamma)$  are the corresponding first-order operators viewed as sections of  $TM \oplus \mathbb{R}$  and  $\Phi$  is the 1-cocycle defined by the projection on  $\mathbb{R}$ . This bracket coincides with (1).

**Proof.** The first part follows immediately from the general result. To show (50) consider  $X = X_1 \wedge X_2$  and  $\mu = \mu_1 \wedge \mu_2$ , where  $X_i \in \mathcal{A}^0$ ,  $\mu_i \in \Omega^{[1]}$ . Then the bracket  $\{\iota_{\mu_1}, \iota_{\mu_2}\}_{X_\Phi^c}$  defined by  $X_\Phi^c$  is given by

$$\langle (X_1)_\Phi^c \wedge (X_2)^v + (X_1)^v \wedge (X_2)_\Phi^c - \iota_\Phi (X_1^v \wedge X_2)^v, d\iota_{\mu_1} \wedge d\iota_{\mu_2} \rangle. \tag{51}$$

This is the linear function

$$\sum_{i,j} (-1)^{i+j} \langle X_i \rangle_\Phi^c (\iota_{\mu_j}) \langle X_{\sigma(i)}^v, \iota_{\mu_{\sigma(j)}} \rangle - \iota_\Phi \langle X^v, d\iota_{\mu_1} \wedge d\iota_{\mu_2} \rangle \tag{52}$$

where  $\sigma$  is the transposition of (1, 2). Taking into account that  $Y_\Phi^c(\iota_\nu) = \iota_{\mathfrak{L}_Y^\Phi \nu}$  and  $Y^v(\iota_\nu) = \iota_{(Y,v)}$  we get that this linear function corresponds to

$$\sum_{i,j} (-1)^{i+j} \langle X_i, \mu_j \rangle \mathfrak{L}_{X_{\sigma(i)}}^\Phi \mu_{\sigma(j)} - \langle X, \mu \rangle \Phi. \tag{53}$$

In view of the properties  $f\mathfrak{L}_Y^\Phi \nu = \mathfrak{L}_{fY}^\Phi \nu - \langle Y, \nu \rangle d^\Phi f$  and  $d^\Phi(fg) = fd^\Phi g + gd^\Phi f - fg\Phi$ , we get (50). To show that this is exactly (1) it suffices to lead easy calculations with  $\mathfrak{L}_{Y+fI}^\Phi \nu = \mathfrak{L}_Y \nu + f\nu$ . □

Of course, instead of the Jacobi structure  $X = \Lambda + I \wedge \Gamma$  we can start from an arbitrary Schouten–Jacobi bracket  $[\cdot, \cdot]^\Phi$  on  $\mathcal{A}(L)$  and an element  $X \in \mathcal{A}^1(L)$  with  $[X, X]^\Phi = 0$  (we will call such  $X$  a *Jacobi element*) and use the formula (50) to define a Lie algebroid structure on  $L^*$  (see [IM1]).

**Example.** Let  $L$  be a Lie algebroid and  $(\Lambda, \Gamma)$  be a *Lie algebroid Jacobi structure*, i.e.  $\Lambda \in \mathcal{A}^1(L)$ ,  $\Gamma \in \mathcal{A}^0(L)$ , satisfy (29). Then, the element  $X = \Lambda + I \wedge \Gamma$  regarded as a bisection of the Lie algebroid extension  $L \oplus_M \mathbb{R}$  relative to the anchor map (cf (30)) is a Jacobi element of the Schouten–Jacobi bracket associated with the 1-cocycle given by  $\Phi((Y, f)) = f$  (cf example 1). Thus, the Poisson lift  $X_\Phi^c$  defines a Lie algebroid bracket on  $L^* \oplus_M \mathbb{R}$  which is formally the same as (1).

The formula (50) for the bracket of ‘1-forms’ generated by  $X_\Phi^c$  can be generalized for arbitrary  $X \in \mathcal{A}^x(L)$  as follows.

**Theorem 14.** *The bracket  $[\mu_0, \dots, \mu_x]_{X_\Phi^c}$  for  $\mu_i \in \Gamma L^*$  induced by  $X_\Phi^c$  by*

$$\iota_{[\mu_0, \dots, \mu_x]_{X_\Phi^c}} = \langle X_\Phi^c, d^\Phi \iota_{\mu_0} \wedge \dots \wedge d^\Phi \iota_{\mu_x} \rangle \tag{54}$$

reads

$$[\mu_0, \dots, \mu_x]_{X_\Phi^c} = \sum_{k=0}^x (-1)^{x+k} \mathfrak{L}_{X_k} \mu_k - x d^\Phi \langle X, \mu_0 \wedge \dots \wedge \mu_x \rangle \tag{55}$$

where

$$X_k = i_{\mu_0 \wedge \dots \wedge \check{\mu}_k \wedge \dots \wedge \mu_x} X. \tag{56}$$

In particular,

$$[d^\Phi f_0, \dots, d^\Phi f_x]_{X_\Phi^c} = d^\Phi \langle X, d^\Phi f_0 \wedge \dots \wedge d^\Phi f_x \rangle. \tag{57}$$

**Remark.** In the case when the differentials  $d^\Phi f$  generate  $L^*$  almost everywhere the formula (57) defines the bracket, thus the homogeneous tensor  $X_\Phi^c$ , uniquely.

Suppose now that  $L$  is a one-dimensional vector bundle over  $M$ . Like in the case of the trivial bundle, the graded space  $ADiff_1(\mathcal{L})$  of first-order multidifferential operators on  $L$  is a Lie subalgebra of the Richardson–Nijenhuis bracket on  $\mathcal{A}(\mathcal{L})$ —the space of multilinear maps of the vector space  $V = \mathcal{L}$  of sections of  $L$ . The difference with the case  $V = C^\infty(M)$  is that we do not have a natural associative algebra structure on  $V$ . However locally, fixing a basic section over an open subset  $N$  of  $M$ , we have an isomorphism of the corresponding graded Lie algebras  $ADiff_1(\mathcal{L}|_N)$  and  $ADiff_1(N)$ . Thus, locally, we have the Poisson lift  $A_\Phi^c$  for any  $A \in ADiff_1(\mathcal{L})$ . The problem is that what corresponds to  $\mathbf{1}$ , and hence what is  $\Phi$  on  $ADiff_1(\mathcal{L})$ , depends on the choice of the local section. However, using the preceding remark, we can conclude that there is a uniquely defined complete lift  $A_{loc}^c$ . This time it is not a multivector field on  $TM \oplus_M \mathbb{R}$  but on the dual  $J_1^*(L)$  to the first jet bundle  $J_1(L)$ . Of course, in the trivial case,  $J_1(L)$  is canonically isomorphic with  $T^*M \oplus \mathbb{R}$ , so  $J_1^*(L)$  is isomorphic with  $TM \oplus \mathbb{R}$  but for non-trivial bundles it is not the case. We just define the complete lift  $A_{loc}^c$  as the unique homogeneous  $(a + 1)$ -vector field on  $J_1^*(L)$  such that

$$\langle A_{loc}^c, dt_{j_1(f_0)} \wedge \dots \wedge dt_{j_1(f_a)} \rangle = j_1(A(f_0, \dots, f_a)) \tag{58}$$

for all sections  $f_0, \dots, f_a$  of  $L$ . Here  $j_1$  means the first jet prolongation of a given section. Since the first jet prolongations of sections of  $L$  generate  $J_1(L)$  over an open-dense subset, the multivector field  $A_{loc}^c$  is uniquely defined. Moreover, it is easy to see that for any local trivialization  $A_{loc}^c$  coincides with  $A_\Phi^c$ . Since all our brackets are local over  $M$ , we get the following.

**Theorem 15.** *The complete lift of first-order differential operators on a one-dimensional bundle defined by (58) is a homomorphism of the Nijenhuis–Richardson bracket on  $ADiff_1(\mathcal{L})$  into the Schouten–Nijenhuis bracket of homogeneous multivector fields on  $J_1^*(L)$ .*

**Corollary 2.** *If  $X \in A^1 Diff_1(\mathcal{L})$  represents a local Lie algebra bracket on  $L$ , then  $X_{loc}^c$  induces a Lie algebroid structure on the first jet bundle  $J_1(L)$ .*

**Remark.** There is no clear analogue of the Jacobi lift for  $ADiff_1(\mathcal{L})$ , since locally it depends strongly on the 1-cocycle  $\Phi$  associated with the trivialization. This suggests that the lift to multivector fields is primary with respect to the Jacobi lift. Note also that  $ADiff_1(\mathcal{L})$  is a  $C^\infty(M)$ -module and the Nijenhuis–Richardson bracket on  $Diff_1(L)$  is a Lie algebroid bracket like in the trivial case. This time, however, we have no splitting  $Diff_1(\mathcal{L}) = Der(\mathcal{L}) \oplus C^\infty(M)$  (it makes no sense) but we have an exact sequence

$$0 \rightarrow C^\infty(M) \rightarrow Diff_1(\mathcal{L}) \rightarrow \mathcal{X}(M) \rightarrow 0 \tag{59}$$

which gives the anchor map for this Lie algebroid and splits only when the bundle  $L$  is trivializable.

## 6. Jacobi bialgebroids

Recall that a *Lie bialgebroid* [KS, MX] is a dual pair  $(L, L^*)$  of vector bundles equipped with Lie algebroid structures such that the differential  $d_*$  induced from the Lie algebroid structure on  $L^*$  as defined by (22) is a derivation of the Schouten–Nijenhuis bracket induced by the Jacobi algebroid structure on  $L$ :

$$d_*[X, Y] = [d_*X, Y] + (-1)^x[X, d_*Y] \quad \text{for all } X, Y \in \mathcal{A}(L). \quad (60)$$

For Jacobi algebroids we will keep formally the same definition.

**Definition.** A *Jacobi bialgebroid* is a dual pair  $(L, L^*)$  of vector bundles equipped with Jacobi algebroid structures such that the differential  $d_*$  induced from the Jacobi algebroid structure on  $L^*$  is a derivation of the Schouten–Jacobi bracket induced by the Jacobi algebroid structure on  $L$ .

**Remark.** Note that the above definition coincides with the definition of *generalized Lie bialgebroid* in [IM1]. Indeed, the condition (4.1) in [IM1] is just (60) for  $X, Y \in \mathcal{A}^0(L)$ , and the condition (4.2) is just (60) for  $Y = \mathbf{1}$ , so every Jacobi bialgebroid is a generalized Lie algebroid in the sense of [IM1]. To show the converse we will use the following lemma.

**Lemma 1.** *If  $d_*$  is the differential on  $\mathcal{A}$  associated with a Jacobi algebroid structure on  $L^*$ , then*

$$\begin{aligned} d_*[X, Y \wedge Z] - [d_*X, Y \wedge Z] - (-1)^x[X, d_*(Y \wedge Z)] \\ = (d_*[X, Y] - [d_*X, Y] - (-1)^x[X, d_*Y]) \wedge Z \\ + (-1)^{x(y+1)}Y \wedge (d_*[X, Z] - [d_*X, Z] - (-1)^x[X, d_*Z]) \\ - (d_*\tilde{D}(X) - \tilde{D}(d_*X) - (-1)^x[X, X_0]) \wedge Y \wedge Z \end{aligned} \quad (61)$$

where  $X_0 = d_*\mathbf{1}$ .

**Proof.** The proof consists of standard calculations using (20) and the following property of the exterior differential:

$$d_*(Y \wedge Z) = (d_*Y) \wedge Z + (-1)^{y+1}Y \wedge (d_*Z) - X_0 \wedge Y \wedge Z. \quad (62)$$

□

**Theorem 16.** *If (60) is satisfied for all  $X, Y \in \mathcal{A}^0(L)$  and all  $X \in V(M) \oplus \mathcal{A}^0(L)$ ,  $Y = \mathbf{1}$ , then it is satisfied in general.*

**Proof.** First, note that

$$d_*\tilde{D}(X) - \tilde{D}(d_*X) - (-1)^x[X, X_0] = d_*[X, \mathbf{1}] - [d_*X, \mathbf{1}] - (-1)^x[X, d_*\mathbf{1}] = 0 \quad (63)$$

for  $X \in V(M) \oplus \mathcal{A}^0(L)$ . Hence, for  $X, Y \in \mathcal{A}^0(L)$  and  $f \in V(M)$  we get from (61) and (60) for elements of  $\mathcal{A}^0(L)$ :

$$(d_*[X, f] - [d_*X, f] - [X, d_*f]) \wedge Y = 0 \quad (64)$$

so (60) is satisfied also for  $X \in \mathcal{A}^0(L)$  and  $Y \in V(M)$ . Using, in turn, this fact when applying  $X = f, Y = g \in V(M), Z \in \mathcal{A}^0(L)$ , to (61), we get

$$(d_*[f, g] - [d_*f, g] + [f, d_*g])Z = 0 \quad (65)$$

so (60) is satisfied for all  $X, Y \in V(M) \oplus \mathcal{A}^0(L)$ . Now, we can prove (60) by induction with respect to the sum  $x + y$  of degrees of  $X$  and  $Y$ . If  $x + y \geq 0$  and, say,  $y > 0$  (the case  $x = y = 0$  is covered by assumption), then we can write  $Y$  as a linear combination of wedge products  $A \wedge B$  with  $a, b < y$  and (60) follows for  $X, Y$  by induction in view of (61). □

**Example.** In [IM1], theorem 5.1, it is shown that any Jacobi element  $X \in \mathcal{A}^1(L)$ ,  $[X, X]^\Phi = 0$  for a Jacobi algebroid structure in  $\mathcal{A}(L)$  associated with a 1-cocycle  $\Phi \in \mathcal{A}^0(L^*)$  gives rise to a Jacobi bialgebroid for which the Lie algebroid structure on  $\mathcal{A}(L^*)$  is given by the formula (50) and the corresponding 1-cocycle is  $-X_\Phi$ . This is, of course, a Jacobi analogue of a triangular Lie bialgebroid in the sense of Mackenzie and Xu [MX].

## 7. Conclusions

We have shown that, by analogy with Lie algebroids (regarded as *odd Poisson brackets*), one can define Jacobi algebroids (regarded as *odd Jacobi brackets*) on the Grassmann algebra  $\mathcal{A}(L)$  associated with a vector bundle  $L$ . Jacobi algebroids in this sense turn out to be objects already studied by Iglesias and Marrero [IM1]. It is possible to develop a Cartan calculus for Jacobi algebroids. We have constructed lifts of tensor fields which transport the Schouten–Jacobi bracket on  $\mathcal{A}(L)$  into the Schouten bracket of multivector fields on the total space  $L$ . This leads to a natural construction which associates a Lie algebroid with every local Lie algebra of Kirillov. We have also shown that a notion of a Jacobi bialgebroid can be consistently introduced with the full analogy to the classical case.

Since every Lie algebroid can be viewed as a particular case of a Batalin–Vilkovisky algebra [Xu], it is natural to look for a similar correspondence in the case of Jacobi algebroids. Initial attempts in this direction have been made in [ILMP]. There is a natural way of defining generating operators for the Schouten–Jacobi brackets and of defining the corresponding homology. We postpone detailed studies of these questions to a separate paper.

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